

## Asymptotically $AdS$ brane black holes

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We study the possibility of having a static, asymptotically  $AdS$  black hole localized on a braneworld with matter fields, within the framework of the Randall and Sundrum scenario. We attempt to look for such a brane black hole configuration by slicing a given bulk spacetime and taking  $\mathbb{Z}_2$  symmetry about the slices. We find that such configurations are possible, and as an explicit example, we provide a family of asymptotically  $AdS$  brane black hole solutions for which both the bulk and brane metrics are *regular* on and outside the black hole horizon and brane matter fields are *realistic* in the sense that the dominant energy condition is satisfied. We also find that our braneworld models exhibit signature change inside the black hole horizon.

PACS numbers: 04.50.+h, 04.70.BW, 11.25.Mj

## I. INTRODUCTION

In [1], using the “squashing” effect a negative bulk cosmological constant has on a four dimensional hypersurface, a brane, Randall and Sundrum (RS) uncovered a mechanism of dynamical localization of gravity. All dimensions, including the one(s) we don’t see, could now be infinitely large and one would still recover, on the brane, the General Relativistic and Newtonian limits of gravity at low energies [2, 3]. It has also been shown that Standard Model matter fields can be constrained on a brane [4] and observations do not rule out braneworlds as cosmological models

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(see [5] for a review). From the astrophysical point of view, both numerical and analytical models of stars have been found [6, 7].

On the other hand, black holes are not well understood in the RS braneworld scenario. The first attempt to find a static black hole solution on the brane was developed in [8] where the 4D Schwarzschild black hole metric on the brane was embedded in a 5D bulk containing an extended singularity: a black string. Were our universe to be a brane in a higher dimensional bulk, such a state of affair is not satisfactory: one might indeed expect astrophysical black holes formed by collapsing matter *on the brane* to be localized on (or at least very close to) the brane (see however [9].) Study of a simple gravitational collapse model [10] on a RS braneworld indicates the difficulty of finding a static vacuum black hole solution localized on a RS braneworld. The difficulties in understanding black hole solutions arise from the fact that in general, brane dynamics generate bulk Weyl curvatures, which then backreact on the brane dynamics. One is then left with the very hard task of solving equations of motion for the coupled system of bulk and braneworld with given suitable initial data. Such a program has not been answered yet, and even numerical approaches are still rather approximative [11]. To simplify the problem, the authors of [12] looked for analytic solutions to the projected Einstein equations on the brane only and found an exact (Reissner-Nordstrom looking) black hole solution. Other similar solutions were subsequently found [13]. These bulks' geometries are not known.

Under such circumstances, it is interesting to ask whether a brane on which a 4D black hole is localized can be found by looking for a slice that intersects a bulk black hole. However, generalizing the work of [8], Kodama showed in [14] that brane solutions with a black hole geometry cannot be found as a slicing of a bulk with  $G(D-2, k)$  symmetry, if the brane is *vacuum* and *not* totally geodesic [32]. In other words if, for simplicity, one wants to keep studying slices of bulks with  $G(D-2, k)$  symmetry to find localized black holes in the RS braneworld scenario—in which the brane is not totally geodesic—one has to look for a *non-vacuum* brane.

Recently, an attempt to find a localized static but *non-vacuum* brane black hole solution as a slice of a  $G(D-2, k)$  bulk was made by Seahra [15]. There, the bulk chosen was the Schwarzschild and Schwarzschild-AdS black hole with spherical three-dimensional geometry, i.e.,  $G(3, k=1)$ , and branes were taken as a planar, asymptotically flat slice of these bulks. Unfortunately these slicing turned out to produce naked singularities with respect to the induced geometry, except when corresponding to the equatorial plane of a bulk black hole, a special case of a totally geodesic brane.

The aim of this paper is to find a regular RS braneworld on which a static, spherically sym-

metric black hole surrounded by realistic matter is localized, by slicing a fixed 5D black hole bulk spacetime. The choice of slicing we will use is motivated by the AdS/CFT-inspired “classical braneworld black hole” vs “quantum black hole” duality of [19] which states: “The black hole solutions localized on the brane in the  $AdS_{d+1}$  braneworld which are found by solving the classical bulk equations in  $AdS_{d+1}$  with the brane boundary conditions, correspond to quantum-corrected black holes in  $d$ -dimensions, rather than classical ones” (see also [18]). Since due to Hawking radiation, black holes in asymptotically flat spacetimes are semi-classically unstable, such a duality would explain the impossibility [10] of finding a static exterior to the Oppenheimer-Snyder collapse of a star in asymptotically flat RS braneworlds (see also [20]). However, asymptotically AdS spacetimes allow (big enough) black holes to be in semi-classical equilibrium [21] with their Hawking radiation. That is the main motivation for turning our attention to the specific slices we study, which are non-vacuum and asymptotically AdS (in a weak sense, see IV). Encouraging results were already obtained in [22]. In this paper we will indeed show with explicit examples that it is possible to construct a localized braneworld black hole surrounded by matter that satisfies the dominant energy condition, when the braneworld is asymptotically AdS (the case of an asymptotically AdS brane black hole as a slice of a bulk black string was studied in [23]).

The plan of the paper is as follows. In the next section, we show that regular slices that cross a bulk black hole horizon can be constructed and we point out why the planar slices of [15] exhibit a curvature singularity there. In Sect. III, we fix our notations and define a bulk slice which is an asymptotically AdS braneworld. We then consider a simple one-parameter family of slices which correspond to asymptotically vacuum and asymptotically AdS braneworlds with black hole horizon, filled with matter satisfying the dominant energy condition. Under our slicing ansatz, we find such braneworlds are possible only for the bulk with three-dimensional spatial geometries corresponding to  $k = 0$  and  $k = -1$ . For the bulk spacetimes with spherical three-dimensional geometry ( $k = 1$ ), our slicing define braneworld with matter violating the dominant energy condition. We also find that some of our braneworlds exhibit an intriguing property: signature change inside brane black holes. In the conclusion, we summarize and discuss our results.

## II. SLICES IN $G(3, k)$ BULK

### A. Bulk spacetime, slicing, and singularity condition

Bearing the Schwarzschild-AdS bulk metric in mind, we consider the following type of five-dimensional static metrics with  $G(3, k)$  spatial symmetry

$$^{(5)}ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + \frac{r^2}{l^2}d\sigma_{(3)}^2, \quad (1)$$

where  $d\sigma_{(3)}^2 = \gamma_{ij}dx^i dx^j$  is the line element of a 3-dimensional space of constant curvature  $k = 0, 1$  or  $-1$ ,

$$d\sigma_{(3)}^2 = dw^2 + l^2 \frac{\sin^2(\sqrt{k}w/l)}{\sqrt{k}} d\Omega_{(2)}^2, \quad (2)$$

with  $d\Omega_{(2)}^2$  being the metric of unit two sphere. For now, we do not impose the Einstein equations so that the function  $A(r)$  is arbitrary.

In [14], Kodama has shown that such bulk metrics cannot embed a *vacuum*, non-totally geodesic brane which describes a static black hole. We would like, in the following, to relax the vacuum condition, and see whether a regular black hole can then be constructed on the brane. As we progress, we will restrict the bulk metric (1) to obtain an asymptotically AdS braneworld with a horizon and positive energy density matter as a slice of such a bulk. It is useful to introduce a new radial coordinate

$$\rho \equiv r \frac{\sin(\sqrt{k}w/l)}{\sqrt{k}}. \quad (3)$$

Then,

$$\frac{dw}{l} = \frac{\rho}{r} \frac{d \log(\rho/r)}{\sqrt{1 - k(\rho/r)^2}}, \quad (4)$$

and the five dimensional metric becomes

$$^{(5)}ds^2 = -A(r)dt^2 + \left[ \frac{1}{A(r)} + \frac{\rho^2/r^2}{1 - k\rho^2/r^2} \right] dr^2 - 2 \frac{\rho/r}{1 - k\rho^2/r^2} dr d\rho + \frac{d\rho^2}{1 - k\rho^2/r^2} + \rho^2 d\Omega_{(2)}^2. \quad (5)$$

Looking for asymptotically AdS branes, we consider slices  $r = r(\rho)$  of the bulk (5). The induced  $4D$  metric on such a slice becomes

$$^{(4)}ds^2 = -A(\rho)dt^2 + \left( \frac{r'^2}{A(\rho)} + B(\rho) \right) d\rho^2 + \rho^2 d\Omega_{(2)}^2, \quad (6)$$

with

$$B(\rho) = \frac{\left(1 - \frac{\rho r'}{r}\right)^2}{1 - \frac{k\rho^2}{r^2}}, \quad (7)$$

where here and hereafter  $r' \equiv dr/d\rho$ .

We are interested in slices that cross a bulk event horizon, so we assume the existence of at least a zero of the function  $A(r)$ . Let  $r_0$  be such that  $A(r_0) = 0$ . We will construct our brane so that there is a  $\rho_0 (> 0)$  which satisfies  $r(\rho_0) = r_0$ . For simplicity we restrict our bulk (1) to those for which  $A(r)$  is at least  $C^2$  except at  $r = 0$ , and  $dA/dr|_{r_0} \neq 0$  (simple zero at  $r = r_0$ , that is, the horizon is non-degenerate). Hence in particular, our bulk spacetimes can be singular only at  $r = 0$ . It is clear that  $\rho = \rho_0$  is also a Killing horizon with respect to the brane metric (6).

The curvature scalars of the metric (6) are all of the form:

$$\begin{aligned} R^\alpha{}_\alpha &= \frac{P_1}{2\rho^2 \left(1 - k\frac{\rho^2}{r^2}\right)^2 \left[r'^2 + A(\rho)B(\rho)\right]^2}, \\ R^{\alpha\beta}R_{\alpha\beta} &= \frac{P_2}{8\rho^4 \left(1 - k\frac{\rho^2}{r^2}\right)^4 \left[r'^2 + A(\rho)B(\rho)\right]^4}, \\ R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} &= \frac{P_3}{8\rho^4 \left(1 - k\frac{\rho^2}{r^2}\right)^4 \left[r'^2 + A(\rho)B(\rho)\right]^4}, \end{aligned} \quad (8)$$

where the numerators  $P$ s are polynomial of  $A, r, \rho$  and their first and second order derivatives. Therefore, possible curvature singularities with respect to the brane metric can happen whenever one of the  $P$ s blows up—which happens only when  $A$  diverges at  $r = 0$ —or when the denominator of (8) becomes zero, that is for instance when

$$\text{a/} \quad \left(1 - k\frac{\rho^2}{r^2}\right) \left[r'^2 + A(\rho)B(\rho)\right]^2 = 0, \quad (9)$$

$$\text{b/} \quad \rho = 0, \quad (10)$$

$$\text{c/} \quad r(\rho) = 0, \text{ if } \lim_{r \rightarrow 0} |A(r)| = \infty. \quad (11)$$

In the slices we will consider below, all the curvature singularities are of one of the above types (they are not mutually exclusive). The case c/ corresponds to 5D bulk singularities. The case b/ would occur even when our bulk spacetime is non-singular, due to a non-trivial embedding. The case a/ also may occur as an embedding singularity, but we need to consider this case with much more care. While singularities of types c/ and b/ are shown to be hidden inside the horizon on the brane  $\rho = \rho_0$ , singularities of type a/ can occur on the brane's horizon  $\rho = \rho_0$ , hence can be a naked singularity for braneworld observers. In fact, from the induced metric (6), one can show

that near the horizon ( $\rho = \rho_0$ ) the Ricci scalar on the brane is

$$R^\alpha{}_\alpha \simeq \frac{\frac{dA}{dr}(r_0)B(\rho_0)}{2r'(\rho_0)^3}. \quad (12)$$

Therefore, in particular, for  $B$  such that  $B(\rho_0) \neq 0$ , there is an embedding curvature singularity where the brane crosses the bulk event horizon if the slice is so that  $r'(\rho_0) = 0$ . We will see below that the naked singularities on the planar slices in [15] are of this type.

It is intriguing to note that although the bulk metric (1) itself is everywhere Lorentzian, the induced metric (6) can be Euclidean inside the event horizon,  $\rho < \rho_0$ , if  $B(\rho)$  can take a sufficiently positive large value so that the  $(\rho, \rho)$  component of the brane metric, eq. (6), can be positive, there. If this is the case, such a brane displays a signature change on a braneworld [24, 25]. It is interesting to notice that in Sect. IV we find such a signature changing slicing under our requirements that branes be asymptotically AdS, regular (at least on and outside the event horizon) and that matter content satisfy the dominant energy condition.

### B. Comparison with earlier results from planar slicing

In [15], Seahra has shown that planar slicing of 5D Schwarzschild and Schwarzschild-AdS bulks are singular as soon as the slices cut the bulk horizon ( $r = r_0$ ). We will now prove that the planar slices examined in [15] correspond to slices for which  $r' = 0$  on the horizon, therefore corresponding to singularities of type  $a/$ .

To find planar slices, introduce the following variable

$$R(r) = \exp \left( \int_{r_1}^r \frac{du}{\sqrt{u^2 A(u)}} \right), \quad (13)$$

where  $r_1$  is a fiducial initial distance, and define the cylindrical coordinates as

$$x = R \sin(w/l), \quad y = R \cos(w/l). \quad (14)$$

The bulk metric is then expressed as

$${}^{(5)}ds^2 = -A(r)dt^2 + \frac{r^2}{x^2 + y^2} \left( dy^2 + dx^2 + x^2 d\Omega_{(2)}^2 \right), \quad (15)$$

where  $r$  is now a function of  $x^2 + y^2$  through eqs. (13) and (14). We therefore find that planar slices correspond to  $y = \text{const}$ , which we differentiate to obtain an  $r = r(w)$  expression for the slice. (We put aside the special cases where  $w$  is constant, which can always be taken to correspond to  $w/l = \pi/2$ .) Using eq. (13), we have

$$0 = \frac{d}{dw} [R(r) \cos(w/l)] = R(r) \left[ \frac{dr/dw}{\sqrt{r^2 A(r)}} \cos(w/l) - \sin(w/l) \right]. \quad (16)$$

For non-degenerate horizons (i.e.,  $A$  has a single zero at  $r = r_0$ ), this implies

$$\frac{dr}{dw} = r \sqrt{A(r)} \tan(w/l) , \quad (17)$$

everywhere. In particular,  $dr/dw|_{r_0} = 0$ . On the other hand, using the coordinate  $\rho$  introduced in eq. (3), we have

$$\frac{dr}{dw} = \frac{\sqrt{1 - k(\rho/r)^2}}{1 - (\rho/r)(dr/d\rho)} \frac{r}{l} \frac{dr}{d\rho} . \quad (18)$$

If for planar slices,  $r \neq \rho$  at the horizon, then  $dr/dw|_{r_0} = 0$  implies  $dr/d\rho|_{r_0} = 0$ . We showed above that such slices possesses an embedding singularity at the horizon, singularity which was analyzed in [15]. If  $\rho = r$ ,  $w/l = \pi/2$  ( $k = +1$ ) (see eq. (3)) so that  $dr/d\rho = 1 \neq 0$ , this corresponds to a totally geodesic brane, equatorial slicing of a Schwarzschild or Schwarzschild-AdS bulk black hole.

### III. UNDERSTANDING THE SLICE $r(\rho)$ AS A BRANE WORLD

In order to establish our notations, we here recall some basic equations used in the RS braneworld scenario, in particular the relation between the extrinsic curvature of a slicing  $\Sigma$  and energy-momentum tensor for matter fields confined on  $\Sigma$ . For simplicity, we assume that our 5D bulk metric  $g_{\mu\nu}$  obeys the vacuum Einstein equations with a negative cosmological constant  $\Lambda_5$ . Assigning surface intrinsic energy momentum tensor  $T_{\mu\nu}$  and a surface tension to  $\Sigma$ , we now think of  $\Sigma$  as a gravitating brane, hence our basic equation is

$${}^{(5)}G_{\mu\nu} = -\Lambda_5 g_{\mu\nu} + \kappa_5^2 \left( T_{\mu\nu} - \frac{6}{\kappa_5^2} \sigma q_{\mu\nu} \right) \delta(\chi) , \quad (19)$$

where  $\kappa_5^2$  denotes 5D gravitational constant, and the constant  $\sigma$  is in proportion to the surface tension. Here  $q_{\mu\nu}$  is the projection tensor (also referred to interchangeably as the induced/brane metric) for  $\Sigma$  defined in terms of the unit normal vector  $n^\mu$  by  $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ , and the function  $\chi$  is introduced to specify the location of  $\Sigma$  as  $\chi = 0$  and  $n^\mu \partial_\mu \chi = \text{const}$  on  $\Sigma$ .

We define the extrinsic curvature of  $\Sigma$  by

$$K_{\mu\nu} = s \frac{1}{2} \mathcal{L}_n q_{\mu\nu} , \quad (20)$$

where  $s = \pm 1$  is introduced for later convenience. Since  $\Sigma$  is a co-dimension one surface, it divides our bulk spacetime into two regions. The ‘orientation’  $s$  decides which side of the bulk, we will consider as the ambient spacetime of  $\Sigma$ . We further assume  $\mathbb{Z}_2$  symmetry with respect to  $\Sigma$  and

take the normal  $n^\mu$  so that when  $s = +1$ ,  $n^\mu$  is directed toward inside the  $\mathbb{Z}_2$  symmetric bulk. Then, integration of eq. (19) along  $\chi$  or the junction condition yields

$$T^\mu{}_\nu = \frac{2}{\kappa_5^2} \left( -K^\mu{}_\nu + (K + 3\sigma)\delta^\mu{}_\nu \right). \quad (21)$$

In order to understand the effective gravitational theory induced on the brane it is useful to consider 4D projected components of eq. (19) [3, 5],

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \mathcal{T}_{\mu\nu}, \quad (22)$$

where  ${}^{(4)}G_{\mu\nu}$  is the four dimensional Einstein tensor on the brane and  $\mathcal{T}_{\mu\nu}$  is the effective energy momentum tensor as seen by a brane observer,

$$\mathcal{T}_{\mu\nu} = \sigma \kappa_5^2 T_{\mu\nu} + \kappa_5^4 \mathcal{S}_{\mu\nu} - \mathcal{E}_{\mu\nu}, \quad (23)$$

where  $\mathcal{E}_{\mu\nu}$  is the projected 5D Weyl tensor, and

$$\mathcal{S}_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\gamma} T^\gamma{}_\nu + \frac{1}{24} q_{\mu\nu} \left( 3 T_{\gamma\delta} T^{\gamma\delta} - T^2 \right), \quad (24)$$

$$\Lambda_4 = \frac{1}{2} (\Lambda_5 + 6\sigma^2). \quad (25)$$

We will denote *real* energy-density and pressures coming from  $T^\mu{}_\nu$  in eq. (21) by

$$T^t{}_t = -\epsilon, \quad T^i{}_i = p_i, \quad (26)$$

and the *effective* energy density and pressures coming from  $\mathcal{T}^\mu{}_\nu$  in eq. (23) by

$$\mathcal{T}^t{}_t = -\tilde{\epsilon}, \quad \mathcal{T}^i{}_i = \tilde{p}_i. \quad (27)$$

#### IV. LOCALIZED BRANE BLACK HOLES IN A SCHWARZSCHILD-ADS BULK

In this section, as our  $G(3, k)$  symmetric bulk spacetime we shall consider 5D Schwarzschild-AdS type metrics (1) with

$$A(r) = k - \frac{2Ml}{r^2} + \frac{r^2}{l^2}, \quad (28)$$

which are solutions to 5D vacuum Einstein equations with a negative constant  $\Lambda_5$  with  $l^2 = 6/|\Lambda_5|$  and  $M$  is related to the mass of the bulk black hole. Note also that for geometries  $k = 0$  and  $k = -1$ , the spatial section of bulk Killing horizons are not compact unless a suitable identification of points along space-like directions is made.



We are looking for a regular asymptotically AdS brane which possesses an event horizon surrounded by matter satisfying some desirable energy conditions (the dominant one). In this paper, we call our brane asymptotically AdS if the asymptotic expansion in  $\rho$  of the first and second fundamental forms of the brane match, to leading order, the asymptotic expansion of a pure AdS brane. In terms of the coordinate system we are employing, our slices are asymptotically AdS of the above sense as soon as

$$q_{tt} \sim \frac{\rho^2}{L^2} + O(1), \quad q_{\rho\rho} \sim \frac{L^2}{\rho^2} + O\left(\frac{L^4}{\rho^4}\right), \quad \rho \rightarrow \infty, \quad (29)$$

and  $-3/L^2 = \Lambda_4$  where  $\Lambda_4$  is defined in eq. (25). The condition on the second fundamental form is then immediately satisfied. This asymptotic AdS condition is slightly different and weaker than the asymptotic AdS conditions of [26], where the metric is required to have asymptotic symmetry group  $SO(2,3)$  in 4 dimensions.

A particularly interesting class of slicing that can be asymptotically AdS in the above sense is given by

$$r = \rho + \gamma, \quad (30)$$

where  $\gamma$  is a constant parameter. We assume that  $\gamma > -\rho_0$ . In the rest of the paper we focus on braneworlds given by the above slicing condition (30). The effective cosmological constant on such branes will be related to this parameter.

### A. Asymptotic energy conditions and singularities

Now that we know what the condition for our slices not to be singular where it crosses the bulk horizon is  $r'(\rho_0) \neq 0$ , we would like to see what are the asymptotic conditions for the slice in order for its matter content not to violate the dominant energy condition near spatial infinity. The dominant energy condition requires energy density to be positive and the absolute value of the radial and angular pressures to be smaller than the energy density.

We first derive an expression for the asymptotic form of the *real* energy density  $\epsilon$  and pressures  $p_i$  for a braneworld defined by (30). Defining the unit normal vector to our slicing  $r = \rho + \gamma$  as

$$n^\mu = g^{\mu\nu} N (\partial_\nu r - r' \partial_\nu \rho), \quad \text{where } N = \frac{1}{\sqrt{A(1 - r' \rho/r)^2 + r'^2(1 - k\rho^2/r^2)}}, \quad (31)$$

and using the formulae (6), (20), (21) and (26) given in the previous sections, we can calculate  $\epsilon$  and  $p_i$ . Since the explicit expressions of the extrinsic curvature  $K^\mu{}_\nu$  and the stress energy tensor

$T^\mu{}_\nu$  themselves are rather complicated and not so illuminating, we do not present them here. The results give, asymptotically,

$$\epsilon = -\frac{3(s\gamma + \sigma l\sqrt{\gamma^2 + l^2(1-k)})}{l\sqrt{\gamma^2 + l^2(1-k)}} + \frac{2s(\gamma^2 + l^2(1-k)^2)l}{(\gamma^2 + l^2(1-k))^{3/2}\rho} + O\left(\frac{l^2}{\rho^2}\right), \quad (32)$$

$$p_\rho = \frac{3(s\gamma + \sigma l\sqrt{\gamma^2 + l^2(1-k)})}{l\sqrt{\gamma^2 + l^2(1-k)}} - \frac{s(\gamma^2(2+k) + 2l^2(1-k)^2)l}{(\gamma^2 + l^2(1-k))^{3/2}\rho} + O\left(\frac{l^2}{\rho^2}\right), \quad (33)$$

$$p_\theta = \frac{3(s\gamma + \sigma l\sqrt{\gamma^2 + l^2(1-k)})}{l\sqrt{\gamma^2 + l^2(1-k)}} - \frac{s(\gamma^2(1+k) + l^2(1-k))l}{(\gamma^2 + l^2(1-k))^{3/2}\rho} + O\left(\frac{l^2}{\rho^2}\right). \quad (34)$$

It is to be noted that the parameter  $M$ , which is proportional to the mass of bulk black hole, plays no role until the next order (see for instance eq. (39) below).

In order to have an asymptotically empty brane, we fine-tune, à la Randall-Sundrum, the vacuum energy related to  $\sigma$

$$\sigma = \sigma_0 \equiv -\frac{s\gamma}{l\sqrt{\gamma^2 + l^2(1-k)}}. \quad (35)$$

The constant term in the above expressions then vanishes. We will keep such a fine-tuning throughout. It is straightforward to see that for the choice  $s = +1$ , the energy density is asymptotically positive for  $k = \pm 1$  and  $k = 0$ . For the energy density to pressures ratios, we asymptotically find:

$$\left|\frac{\epsilon}{p_\rho}\right| = \left|\frac{2(\gamma^2 + l^2(1-k)^2)}{\gamma^2(2+k) + 2l^2(1-k)^2}\right| + O\left(\frac{l}{\rho}\right), \quad (36)$$

$$\left|\frac{\epsilon}{p_\theta}\right| = \left|\frac{2(\gamma^2 + l^2(1-k)^2)}{\gamma^2(1+k) + l^2(1-k)^2}\right| + O\left(\frac{l}{\rho}\right). \quad (37)$$

So,

$$\begin{aligned} & > 1 && \text{for } k = -1 \text{ and } \gamma \neq 0, \\ \left|\frac{\epsilon}{p_\rho}\right| \rightarrow & = 1 && \text{for } (k = 0) \text{ or } (\gamma = 0 \text{ and } k \neq 0), \\ & = 2/3 < 1 && \text{for } k = 1. \end{aligned} \quad (38)$$

For the dominant energy condition to be satisfied asymptotically, we thus have to rule out the  $k = 1$  case. It is easy to see that  $|\epsilon|/|p_\theta| > 1$ , except when  $\gamma = 0$  and  $k = 1$ , which is a totally geodesic brane ( $K_{\alpha\beta} = 0$ ). We will not consider that case any further. The  $(k = 0)$  case requires a next order analysis, and we find

$$\left|\frac{\epsilon}{p_\rho}\right| = 1 - \frac{2\gamma^3 M}{l^2(\gamma^2 + l^2)} \frac{l^3}{\rho^3} + O\left(\frac{l^4}{\rho^4}\right), \quad (39)$$

so that in the  $k = 0$  case, the dominant energy condition (DEC) is only satisfied asymptotically if  $\gamma < 0$  (corresponding to a brane with positive vacuum energy).

Recapitulating (and completing) the asymptotic behavior of the real matter, we find (“*Yes*” means DEC satisfied asymptotically and “*No*” means DEC violated)

	$k = -1$	$k = 0$	$k = 1$
$\gamma > 0$	<i>Yes</i>	<i>No</i>	<i>No</i>
$\gamma = 0$	<i>Yes</i>	<i>Yes</i>	–
$\gamma < 0$	<i>Yes</i>	<i>Yes</i>	<i>No</i>

(40)

Note that the  $k = 1$ ,  $\gamma = 0$  case corresponds to a  $w/l = \pi/2$ ,  $r = \rho$  slice. For such a slice, it is straightforward to obtain the induced metric from (1), and it corresponds to a totally geodesic brane.

Thus, from the above observations of braneworld stress-energy, physically interesting cases are those in which the bulk has a spatial geometry of either  $k = 0$  or  $k = -1$ , and a slicing  $\Sigma$  is given by eq. (30) with  $\gamma \leq 0$ , and the bulk orientation is  $s = +1$ . We note that for  $k = 0$  and  $k = -1$ , the bulk event horizon is no longer spatially compact. On the other hand, our spherically symmetric slicing  $\Sigma$  intersects the bulk event horizon at  $\rho = \rho_0$ , and the region  $\rho > \rho_0$  of  $\Sigma$  always is outside the bulk event horizon. Therefore  $\Sigma$  divides the bulk black hole horizon into two parts: (i) a spatially compact portion,  $\{r = r_0, 0 \leq \rho \leq \rho_0\}$ , which is spherically symmetric, and (ii) its complement,  $\{r = r_0, \rho_0 < \rho < \infty\}$ , which is infinitely extended in the bulk. Then, depending upon the choice of  $s = \pm 1$  (i.e., which side of the bulk we discard before taking  $\mathbb{Z}_2$  symmetry), the resultant  $\mathbb{Z}_2$  symmetric bulk will contain either a spatially compact portion of the bulk event horizon, or an infinitely extended portion of the bulk horizon. Our choice  $s = +1$  is indeed the former case. In fact, noting that the coordinate components of the normal vector, eq. (31), are

$$n^r = -NA \frac{-\gamma}{r}, \quad n^\rho = -NA \left\{ \frac{-\gamma \rho}{r^2} + \frac{1}{A} \left( 1 - k \frac{\rho^2}{r^2} \right) \right\}, \quad (41)$$

one can easily see that  $n^\rho < n^r$  outside the event horizon and hence that  $n^\mu$  is directed toward the region  $\{r > \rho + \gamma\}$ , as depicted in fig. (1). With the choice  $s = +1$ , this region corresponds to our  $\mathbb{Z}_2$  symmetric bulk spacetime. We would like to emphasize that this choice  $s = +1$  is the case we consider in order to satisfy the energy condition for real matter. By this construction, the resultant  $\mathbb{Z}_2$  symmetric 5D spacetime contains a RS brane as a boundary and a spatially compact portion of the 5D event horizon attached to the brane. This black hole looks like a compact object from both the bulk metric and the brane’s intrinsic metric view points, hence one can interpret this geometry as a black hole localized on the brane.

Our braneworld may change the signature of its induced metric, despite the fact that the bulk spacetime is everywhere Lorentzian. In fact, by inspecting  $B(\rho)$  of eq. (7), we find that for  $k = -1, 0$

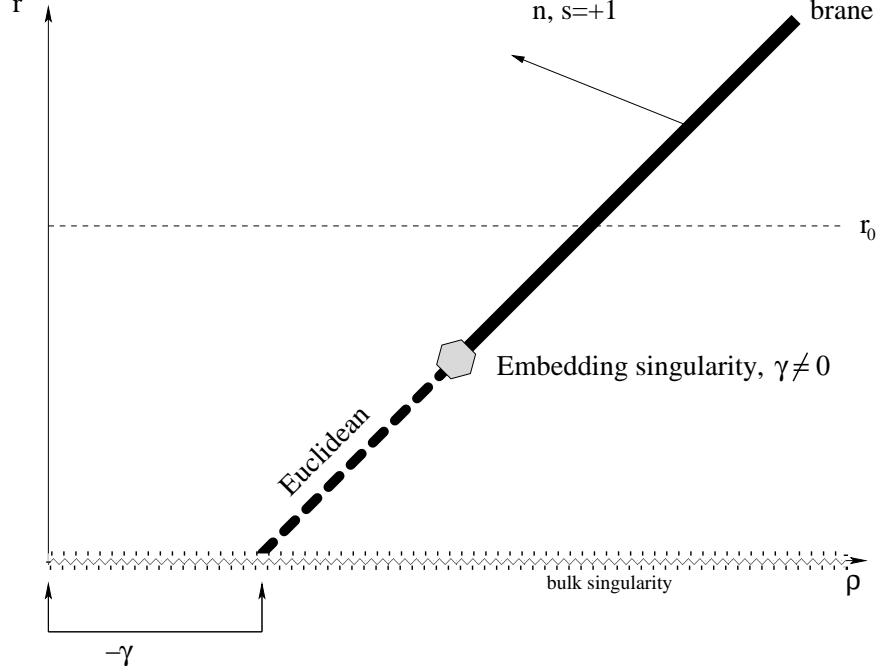


FIG. 1: Schematic representation of the slice  $r = \rho + \gamma$  in the  $k = 0, -1$  bulk. For  $s = +1$ , the upper side of the thick line (brane) corresponds to our bulk under  $\mathbb{Z}_2$  symmetry. Our brane cut off a compact portion from the bulk event horizon (dashed horizontal line). Note that the 2-dimensional subspace spanned by  $(r, \rho)$  is Riemannian above the line  $r = r_0$  but is Lorentzian below the line.

whenever  $\gamma \neq 0$ , our braneworlds display such a signature change inside the event horizon. When  $\gamma = 0$ , no signature change happens for any  $k$ . (Note that for  $k = +1$ , the signature change can occur inside the horizon if  $\gamma > 0$ , but if  $\gamma < 0$ , it can occur even outside the horizon, at large  $\rho$ .) Furthermore, we can see that the brane's intrinsic geometry becomes singular at signature changing points. Such signature changes have been studied e.g. in [25]. Indeed, we find from eq. (6) that in general the braneworld signature change can occur when  $q_{tt}q_{\rho\rho} = -\{r'^2 + A(\rho)B(\rho)\}$  changes its sign, where  $q_{\mu\nu}$  is the induced metric, eq. (6). It then immediately follows from (9) that the associated singularities are of type a/.

### B. Black hole on the brane

Here we study the character of the stress energy on our braneworlds,  $\epsilon$  and  $p_i$ , in more detail. As a typical case we focus on the  $\gamma < 0$  case. The analysis of the dominant energy condition above

restricts us to the  $k = -1$  case. The induced 4D metric is

$${}^{(4)}ds^2 = - \left( -1 - \frac{2Ml}{(\rho + \gamma)^2} + \frac{(\rho + \gamma)^2}{l^2} \right) dt^2 + \left[ \frac{1}{-1 - \frac{2Ml}{(\rho + \gamma)^2} + \frac{(\rho + \gamma)^2}{l^2}} + \frac{\gamma^2}{(\rho + \gamma)^2 + \rho^2} \right] d\rho^2 + \rho^2 d\Omega_{(2)}^2 \quad (42)$$

One can immediately see that there is an event horizon on the brane where the brane intersects the bulk event horizon. We now have a look at what this horizon hides. The general analysis in Sect. II shows that in the  $k = -1$  case we are considering here, since  $r' = 1$  and  $A(\rho)$  has no zero outside the horizon, there can be no curvature singularity outside the horizon. Indeed, the Ricci, Ricci square and Riemann square curvature scalars for the induced metric (42) can be written as before eq. (8). Explicitly, curvature singularities of the 4D metric (42) occur when

$$\text{a/} \quad -1 - \frac{2Ml}{(\rho + \gamma)^2} + \frac{(\rho + \gamma)^2}{l^2} = -\frac{(\rho + \gamma)^2 + \rho^2}{\gamma^2} ; \quad (43)$$

$$\text{b/} \quad \rho = 0 ; \quad \text{or} , \quad (44)$$

$$\text{c/} \quad \rho = -\gamma . \quad (45)$$

However, since the bulk black hole horizon, from the brane point of view is located at  $\rho = \rho_0$ , where

$$\rho_0 = \sqrt{l^2/2 + \sqrt{l^4 + 8Ml^3}/2} - \gamma , \quad (46)$$

the curvature singularities on the brane are hidden by the horizon for all the cases a/, b/, c/ .

At infinity, the induced metric can be expanded in  $\rho/l$ , giving

$$q_{tt} \sim -\frac{\rho^2}{l^2} , \quad q_{\rho\rho} \sim \frac{l^2 + \gamma^2/2}{\rho^2} . \quad (47)$$

Re-scaling the time coordinate  $t \rightarrow \frac{l}{L}t$  with  $L \equiv \sqrt{l^2 + \gamma^2/2}$ , one can check that (47) has the same asymptotic behavior at the brane's infinity  $\rho \rightarrow \infty$  as the  $AdS_4$  metric with cosmological constant  $\Lambda_4 \equiv -3/L^2$  corresponding to eq. (25), that is

$$\Lambda_4 = -\frac{6}{2l^2 + \gamma^2} . \quad (48)$$

One can also see the above result from the expansion of the Ricci scalar

$$R^\alpha{}_\alpha = -\frac{3}{L^2} - \frac{6\gamma(8l^2 + \gamma^2)}{L(2l^2 + \gamma^2)^2} \frac{L}{\rho} + O\left(\frac{L^2}{\rho^2}\right) . \quad (49)$$

Although in principle we have all the ingredients to obtain analytically all the characteristics of these black hole solutions on the brane, in the case of asymptotic non-totally geodesic brane  $\sigma \neq 0$ , the expression for the stress tensor and the curvatures are quite involved. We therefore

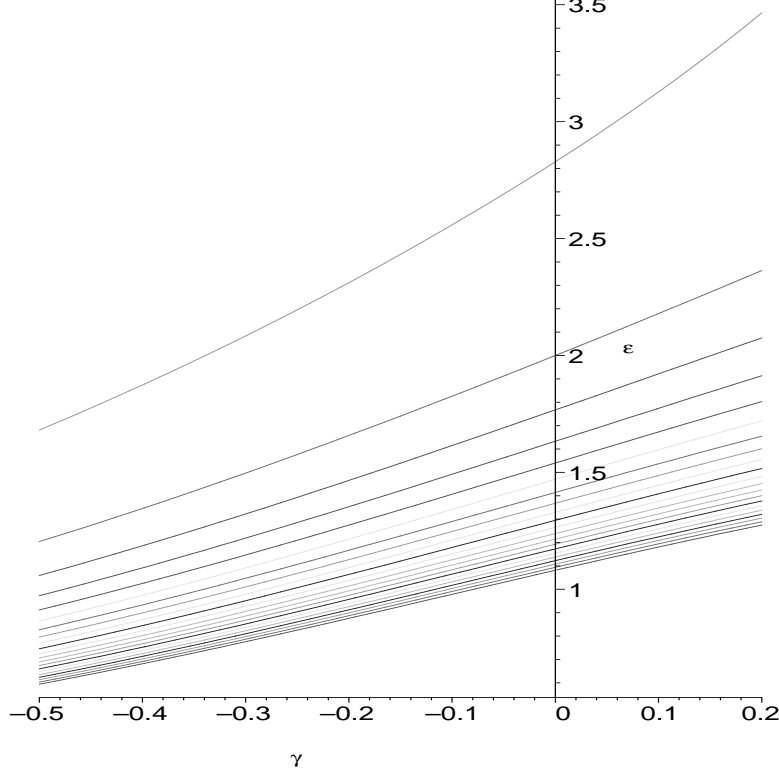


FIG. 2: Values of the real energy density at the horizon for  $\gamma \in [-0.5, 0.2]$  for  $M = 0 \dots 20$ . The upmost plot corresponds to  $M = 0$ , the lowest one to  $M = 20$ .

verify numerically that many of our slices are explicit examples of a regular brane localized black hole with surrounding matter that fulfill the dominant energy conditions *everywhere* outside the brane's Killing horizon. We here set  $l = 1$  and consider dimensionless variables.

In fig. (2), the real energy density  $\epsilon$  can be seen to be positive at the horizon for a wide range of slices. It can be shown that the parameter,  $M \propto \{\text{the mass of the 5D bulk black hole}\}$ , needs to be above a threshold  $M_0 \approx 0.15$  for all the negative values of  $\gamma$  to give a positive energy density at the horizon. Unless otherwise stated, we will only consider such bulks in what follows. For the dominant energy condition, we find that at the horizon,  $|\epsilon/p_\rho| = 1$ , and in fig. (3), we plot  $|\epsilon/p_\theta|$  for a few values of  $M$  from 1 to 20. For these, the dominant energy condition is satisfied for  $M$  small enough at least for  $\gamma \in [-0.15, 0]$ , as is illustrated by the plots. It can be shown that for large  $-\gamma/l$ ,  $|\epsilon/p_\theta| < 1$  (so the dominant energy condition is violated).

For clarity, we now consider an explicit example of slice:  $M = 2$  and  $\gamma = -0.1$ . The dominant energy condition is satisfied everywhere outside the horizon  $\rho_0 \approx 1.70$ , as is illustrated in figs. (4, 5, 6).

Other values of  $M$  and  $\gamma$  could have been chosen. For our particular choice, there is a singularity

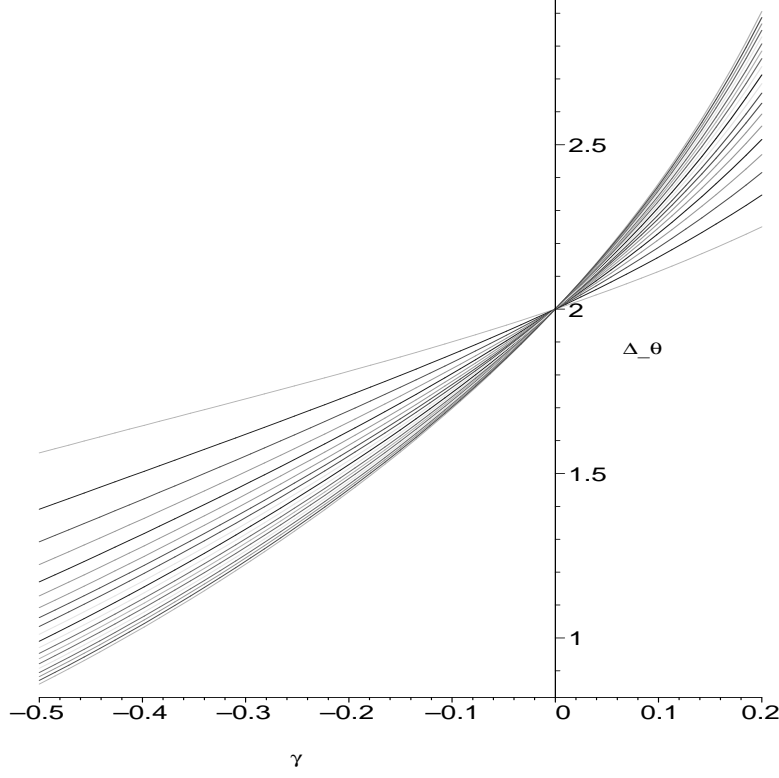


FIG. 3: Values of  $\Delta_\theta = |\epsilon/p_\theta|$  at the horizon for  $\gamma \in [-0.5, 0]$  for  $M = 1 \dots 20$ .

of type  $a/$  at  $\rho = \rho_a \approx 0.68$ , where our brane changes signature. Assuming the slice can be continued past this first singularity, there is another curvature singularity (of type  $c/$ ) at  $\rho = \rho_c = 0.1$ .

To conclude this section we would like to see what happens asymptotically for the effective stress tensor. We have

$$\tilde{\epsilon} = \frac{4\gamma(\gamma^2 + 4l^2)}{l(2l^2 + \gamma^2)^2} \frac{l}{\rho} + O\left(\frac{l^2}{\rho^2}\right), \quad (50)$$

$$\tilde{p}_\rho = -\frac{2\gamma(\gamma^2 + 8l^2)}{l(2l^2 + \gamma^2)^2} \frac{l}{\rho} + O\left(\frac{l^2}{\rho^2}\right), \quad (51)$$

$$\tilde{p}_\theta = -\frac{8l^2\gamma}{l(2l^2 + \gamma^2)^2} \frac{l}{\rho} + O\left(\frac{l^2}{\rho^2}\right). \quad (52)$$

Until now, in order to have a positive vacuum energy on our brane, we imposed on  $\gamma$  to be negative. We here see that this constrains the effective matter to violate the weak energy condition as soon as  $\rho$  is large enough. Had we chosen our brane to have a negative or null vacuum energy, the effective matter would have had a positive energy density asymptotically and it is easy to see that it would also have satisfied the dominant energy condition. It is to be noted that particular slices with  $\gamma > 0$  can be found such that the real matter satisfies the dominant energy condition everywhere outside the horizon as before. Such an example is given by  $(M = 2, \gamma = 0.1)$ . The slice  $\gamma = 0$  for

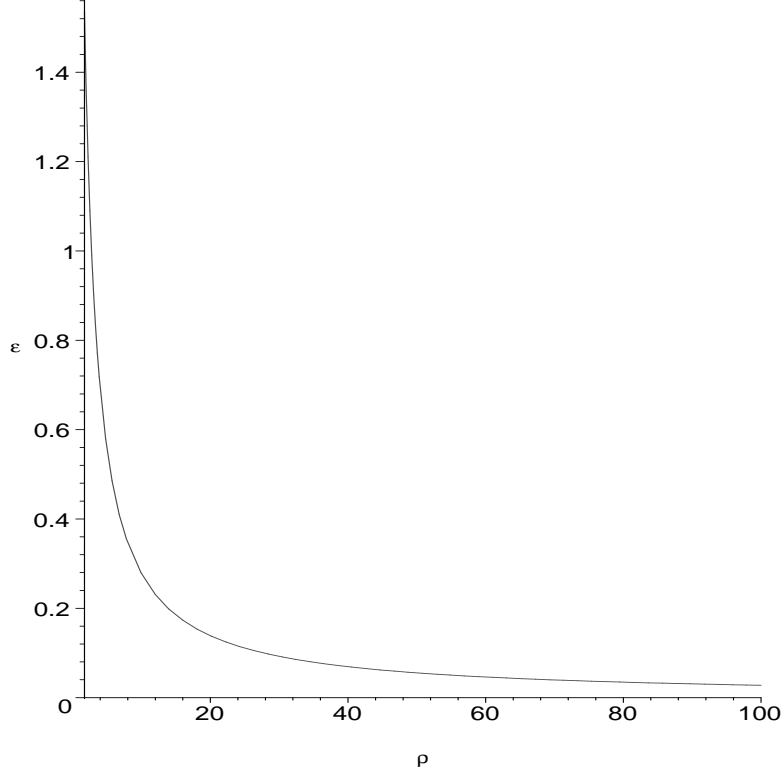


FIG. 4: The real energy density on the brane is seen to be positive everywhere outside the horizon. We have taken the particular values  $M = 2$  and  $\gamma = -0.1$  but the qualitative behavior is the same for  $0.2 \lesssim M \lesssim 10$  and  $0 \leq -\gamma \lesssim 0.15$ .

$k = -1$  or  $0$  (generalizing the  $k = 1$  “equatorial slice”) allows us to treat the problem analytically, this will be the topic of the next section.

### C. Black hole in asymptotically totally geodesic brane

In this section, we concentrate on the  $\gamma = 0$  case, with  $k \leq 0$ , which implies  $\sigma = 0$ . Since real and effective matter die off at spatial infinity in the braneworld, the extrinsic curvature of such a brane vanishes at spatial infinity. This makes the brane asymptotically totally geodesic.

When  $\sigma = 0$ , the real matter fields (i.e., source term linear to  $T_{\mu\nu}$ ) are decoupled from the brane gravity (see eqs. (22) and (23)). In this case the only source for the brane gravity through eq. (22) are  $\mathcal{S}_{\mu\nu}$  and  $\mathcal{E}_{\mu\nu}$  tensors coming, respectively, from the junction conditions and the projection of the 5D Weyl tensor onto the brane. We incidentally note that this limit corresponds in the holographic picture of [18, 19, 20, 27] to the situation in which our black hole is surrounded by “quantum” matter only.



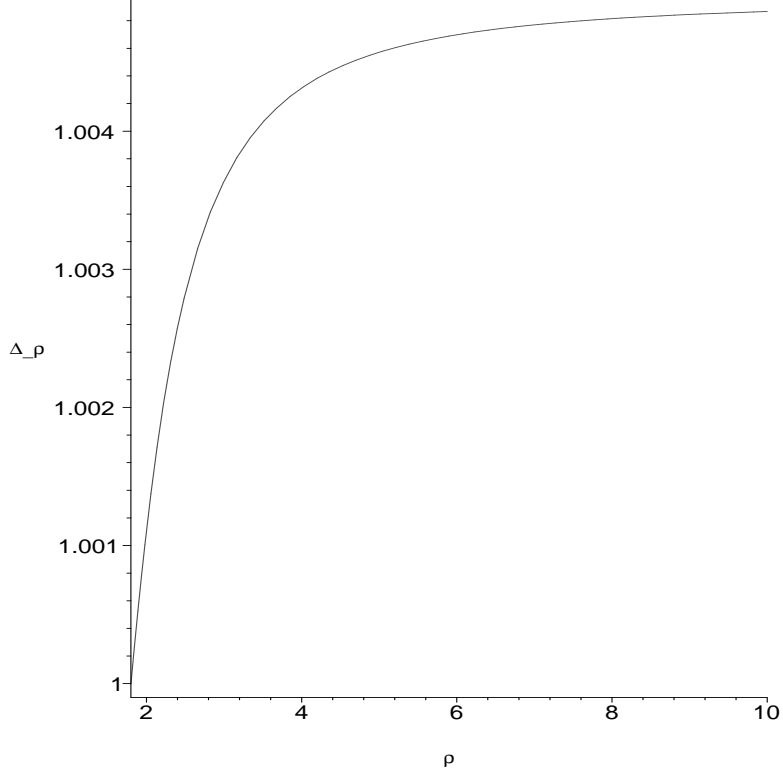


FIG. 5:  $\Delta_\rho = |\epsilon/p_\rho|$  on the brane is seen to be larger than 1 everywhere outside the horizon. We have taken the particular values  $M = 2$  and  $\gamma = -0.1$  but the qualitative behavior is the same for  $0.2 \lesssim M \lesssim 10$  and  $0 \leq -\gamma \lesssim 0.15$ .

These slices provide explicit analytical examples of spherically symmetric, localized brane black holes where the surrounding matter does not violate dominant energy conditions anywhere outside the horizon. In these black hole solutions, there is only one singularity which is a central singularity hidden by a Killing horizon. In fact, the induced metric on the brane is

$$^{(4)}ds^2 = - \left( k - \frac{2Ml}{\rho^2} + \frac{\rho^2}{l^2} \right) dt^2 + \frac{d\rho^2}{k - \frac{2Ml}{\rho^2} + \frac{\rho^2}{l^2}} + \rho^2 d\Omega_{(2)}^2, \quad \text{with } k = -1, 0. \quad (53)$$

So, the Killing horizon is located at

$$\rho_0 = \frac{l}{2} \sqrt{-2k + 2\sqrt{1 + 8M/l}}. \quad (54)$$

The matter content on the brane is

$$\begin{aligned} \epsilon &= \frac{2\sqrt{1-k}}{\rho} = -p_\rho = -2p_\theta = -2p_\phi \\ \tilde{\epsilon} &= (1-k) \frac{\rho^2 - Ml}{\rho^4} = -\tilde{p}_\rho, \\ \tilde{p}_\theta &= \tilde{p}_\phi = -(1-k) \frac{Ml}{\rho^4}, \end{aligned} \quad (55)$$

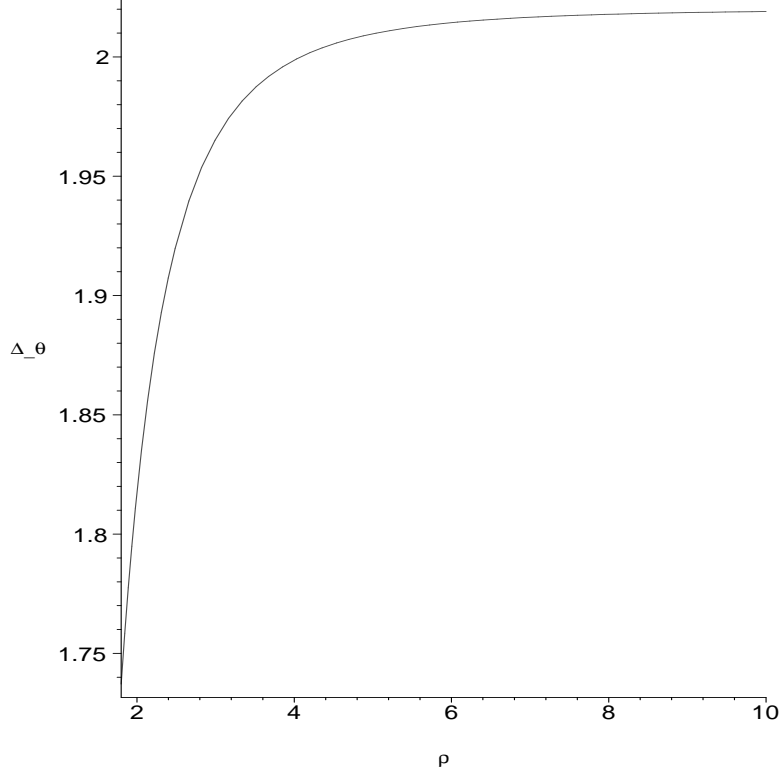


FIG. 6:  $\Delta_\theta = |\epsilon/p_\theta|$  on the brane is seen to be larger than 1 everywhere outside the horizon. We have taken the particular values  $M = 2$  and  $\gamma = -0.1$ , but the qualitative behavior is the same for  $0.2 \lesssim M \lesssim 10$  and  $0 \leq -\gamma \lesssim 0.15$ .

where the “ $M$ ” contribution comes from the projection of the Weyl tensor onto the brane and the Einstein tensor is diagonal.

In the case of big black hole mass  $M \gg l$ , it is noticeable that the energy density for the effective matter, although positive asymptotically, becomes negative when approaching the black hole horizon (when  $\rho \leq \sqrt{Ml}$ ). This is a local effect, and when we consider the effective matter to correspond to quantum matter [18, 19, 20, 27], this is reminiscent of semi-classical considerations in asymptotically flat black hole spacetimes [28, 29]. The holographic interpretation can be trusted only in this limit [18]. It is easy to calculate the curvature scalars. They are everywhere regular except at  $\rho = 0$ . For instance,

$$R^\alpha{}_\alpha = -\frac{12}{l^2} + \frac{2(1-k)}{\rho^2}, \quad (56)$$

and  $R_{\alpha\beta}R^{\alpha\beta} \propto \rho^{-8}$ ,  $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto \rho^{-8}$  near  $\rho = 0$ . We therefore have a central space-like singularity located at  $\rho = 0$ , provided  $M > 0$ . (When  $k = -1$ , even for  $M < 0$  the bulk spacetime possess an event horizon that hides a time-like bulk singularity, provided  $-1 < 8M/l$ . Accordingly the brane’s singularity also is time-like, which is hidden inside the brane black hole horizon.)

The zero mass bulk black hole case,  $M = 0$ , is special: In this case, our bulk is locally pure  $AdS_5$ , hence is regular everywhere. With the coordinate choice  $k = -1$ ,  $AdS_5$  bulk has a (non degenerate) Killing horizon at  $r = 0$ . Our brane (53) (with  $M = 0$ ,  $k = -1$ ) then describes a black hole with a Killing horizon at  $r = \rho = l$ . The central curvature singularity of this brane, due to this non-trivial embedding, is space-like and hidden by the Killing horizon. On the other hand, the choice  $k = 0$  corresponds to the horospherical coordinates of  $AdS_5$  bulk, which now has a Killing horizon at  $r = 0$  of this coordinate. Our brane in this case hits this bulk Killing horizon at  $\rho = 0$ , but now the brane becomes singular on that horizon. The dominant energy conditions are nowhere violated for both effective and real matter.

## V. CONCLUSION

One of the most important unsolved problems in braneworld scenario is probably the missing localized black hole solution on a braneworld. Until now, only negative results of (vacuum) localized braneworld black holes had been put forward. Motivated by the conjecture that *localized black hole on a Randall-Sundrum brane* holographically corresponds to a *semiclassical four dimensional black hole* [18, 19], we tried to find a non-vacuum, asymptotically AdS black hole solution on such a (non totally geodesic) brane.

To achieve this goal, we hunted for possible slices of the Schwarzschild-AdS bulk which cut the bulk black hole horizon producing a smooth horizon on the brane. We showed that this is possible if one introduces suitable matter on the brane, solely determined by the junction conditions on the brane itself. Requiring such matter to be realistic, we looked for slicing corresponding to a brane filled by matter satisfying the dominant energy conditions.

More explicitly, we studied the simplest one-parameter family of slices which obeyed these constraints. Although our parameter turned out to be constrained for our energy conditions to be satisfied (outside the horizon), we found a whole range of values for which our slices correspond to a (regular) localized “black hole” on a brane. In some particular cases, corresponding to a generalization of the “equatorial slice” of the spherical Schwarzschild AdS bulk to hyperbolic and flat three-dimensional geometries, we found explicit analytical solutions with a horizon hiding a single point-like singularity at the center. We also noticed that for non-zero large bulk black hole mass, the energy conditions for the effective matter are satisfied outside a spherical region surrounding the black hole horizon. This is reminiscent of semi-classical results obtained in asymptotically flat black hole spacetimes (see [28, 29] and references therein). For zero mass bulk black hole case,

both the real and effective energy conditions are satisfied everywhere outside the horizon.

We also showed that it is possible for the part of our braneworlds that is hidden by the Killing horizon to undergo a signature change. From the braneworld view point, the appearance of the Euclidean signature region might be interpreted in the quantum theoretical context, such as Euclidean quantum gravity on the braneworld. On the other hand, the bulk spacetime is everywhere Lorentzian, hence one may expect that the braneworld signature change could entirely be understood in terms of bulk classical theory. Such an expectation is in accord with the spirit of a holographic idea in the sense that quantum phenomena on braneworld have some correspondence to bulk classical phenomena. However, whether such a signature changing braneworld can be realized as a solution of a well-posed initial value problem (e.g., 5D Einstein equations with suitable initial data) is a non-trivial question.

By construction, our results fit in a Randall-Sundrum braneworld scenario. We nevertheless believe that our solutions can be of phenomenological importance beyond that framework in understanding real astrophysical or microscopic black holes if extra dimensions are part of an ultimate theory of gravity.

Finally, from the perspective of the conjecture of [18, 19], we would like to conclude by pointing out that the brane black hole solutions found in this paper are, to our knowledge, neither known semi-classical solutions nor possibly obtained by perturbations thereof (see [22] for further similar comments). We found that black holes satisfying energy conditions are possible only in the case  $\Lambda_4 \simeq \Lambda_5$ . In this regime, four dimensional gravity is not localized at least at spatial infinity [30]. Therefore, although our results apparently contradict the conjecture of [19] and [18], it is not clear whether our black hole solutions can be used for this holographic conjecture [33] as, at least at spatial infinity, we would expect our black hole solutions to correspond to a deformed conformal field theory without gravity [31]. Clarification of this problem is beyond the scope of the present work.

### Acknowledgments

The authors are indebted with M. Sasaki for enlightening discussions. We also wish to thank S. Hartnoll, S. Hawking, R. Maartens and G. Procopio for useful comments and discussions. AI was supported by NSF grant PHY 00-90138 to the University of Chicago. CG was supported in part by JSPS fellowship (JSPS/FF1/412) and in part by PPARC research grant (PPA/P/S/2002/00208).

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  - [33] We thank Takahiro Tanaka for pointing this out to us.